Integrated Vehicle Stability Control Using the Principles of Optimal Tracking

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Abstract

This paper applies optimal tracking control to vehicle chassis systems integration for stability control of a passenger vehicle. Yaw moment control using an active front steer system and differential braking is described using this approach. The resulting integrated vehicle stability control system is capable of enhancing the vehicle stability. Simulation results of a vehicle with integrated control of these systems are compared to a vehicle which has these systems acting independently. The results show that an integrated control of these systems can achieve improved performance compared to a vehicle with independently controlled systems.

Nomenclature

C_f = Front axle cornering stiffness, N/rad.

C_r = Rear axle cornering stiffness, N/rad.

a = Distance from front axle to vehicle center of gravity, m.

b = Distance from front axle to vehicle center of gravity, m.

I_{zz} = Moment of inertia about vehicle z-axis, kg-m^2.

M_B = Vehicle mass, kg.

δ = Front road wheel angle, radians.

F_f = Front lateral force, N.

F_r = Rear lateral force, N.

α_f = Front slip angle, radians.

α_r = Rear slip angle, radians.

V_y = Lateral velocity, m/s.

V_x = Longitudinal velocity, m/s.

δ_f = front road wheel angle, radians

δ_r = rear road wheel angle, radians

1. Introduction

Vehicle stability control systems have been shown to increase vehicle passenger safety by well over 30% [Farmer, 2004]. The design of such systems started with direct yaw control of vehicle brake systems [Ghoneim 2000] and are now becoming standard on many production vehicles [GM press release 2005]. More recently increased actuation on vehicles, such as adaptive dampers and active steering systems, has enabled greater control over the vehicle dynamic characteristics. These systems were initially treated as stand alone devices from the perspective of control. For instance, active suspension systems to maximize ride comfort [Howell 1997, Frost 1996] , and ABS Systems to minimize stopping distance [Jun 1998, Petersen 2001, Johansen 2001]. It has however long been recognized that the integrated control of these separate systems can achieve greater overall performance than the systems acting independently [Fruechte 1998, Mastinu 1993, Roppenecker 1993, Tomizuka 1995, Nobe 2001].

The integration of an active suspension system and slip control via braking was investigated by Smakman [Smakman 2000] to improve the lateral vehicle dynamics. The integration of suspension with steering has also been considered by Harada [Harada 1999] and [Hirano 1993] who investigated the application of $H\infty$ control to the integration of four wheel steer and suspension. The integration of braking and steering has been considered by Salman [Salman 1990] ,and also by Nagai [Nagai 1998] who examined the effects of active front steer and rear wheel steer vehicles [Nagai 2002]. Several authors have also considered combining steering, suspension and all wheel drive on a vehicle, for instance, [Sato 1993, Horiuchi 1999].

In this paper an integrated vehicle stability system is developed for combining braking and active front steering using the principles of optimal tracking control. The approach taken in this paper is to illustrate the approach using a single track ('bicycle model') vehicle model. The assumption of the availability of the necessary state feedback is made. Sensors are typically available for measuring the vehicle lateral acceleration and the yaw rate but not for the lateral velocity, this needs to be estimated. The problem of state estimation, while an important one from an implementation standpoint, is beyond the scope of this paper. Various authors have previously considered observers and state estimators for the vehicle lateral velocity [Best 2000, Ungoren 2002] ,and also considered using GPS to improve the estimation accuracy [Ryu 2002].

The paper is organized as follows. The yaw-plane vehicle model with terms for the active front steer and braking moment is first introduced in section 2. The next section outlines the principles of optimal tracking control. Section 4 then adapts the vehicle model so that it is suitable for optimal tracking control and describes the model to be tracked. Section 5 then shows some illustrative simulations of the integration of active front steer and braking and compares these to situations where the vehicle systems are controlled independently and are acting in a stand-alone manner. A discussion of the results of this approach is then undertaken. The appendix contains some of the basic mathematics relating to optimal tracking control.

2. Bicycle Model with Control Augmentation Terms

The yaw and lateral dynamics of a vehicle undergoing handling maneuvers can be characterized in its simplest form by a yaw-plane or bicycle model [Gillespie 1992, Kiencke 2000]. The bicycle model is a simplified two-state linear model of the vehicle lateral dynamics and is obtained from figure 1 and the following two equations.

$$M\left(\dot{V}_y + V_x r\right) = F_f + F_r$$
$$I_{zz}\dot{r} = aF_f - bF_r + \Delta\Psi_{braking}$$

Where

$$Ff = C_f\alpha_f = C_f\left(-\frac{V_y + ar}{V_x} + \delta_f + \Delta\Psi_{Front\_Steering}\right)$$
$$Fr = C_r\alpha_r = C_r\left(-\frac{V_y - br}{V_x}\right)$$

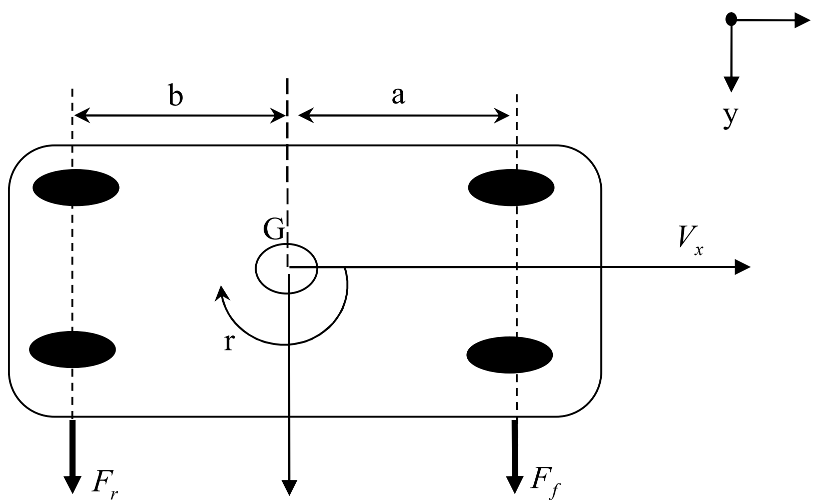

Figure 1. Vehicle Free Body Diagram

Linear approximations of the vehicle tire forces are used that are valid for small slip angles. In state-space formulation, the two-state system takes the form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_{11}\\ b_{21}\end{bmatrix}\begin{bmatrix}\delta_f\\ \delta_r\end{bmatrix} + \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}\Delta\Psi_{Front\_Steering}\\ \Delta\Psi_{Braking}\end{bmatrix}$$

The two states are the lateral velocity and the yaw rate, so we define:

$x_1 = V_y$ and $x_2 = r$.

where δ_f = front road wheel angle. $\Delta\Psi_{Front\_Steering}$ is the front steering yaw moment correction term that can be mapped to the augmented front steer angle using the following equation:

$$\Delta\Psi_{Front\_Steering} = aC_f \Delta\delta_f$$

Where $\Delta\delta_f$ is the augmented front steer angle. $\Delta\Psi_{Braking}$ is the yaw moment correction from differential braking between the left and right wheels. The state transition matrix elements are computed from vehicle parameters and vehicle longitudinal velocity V_x and are defined below:

$$a_{11} = -\frac{C_f + C_r}{M_B V_x} \qquad a_{12} = \frac{-aC_f + bC_r}{M_B V_x} - V_x$$

$$a_{21} = \frac{-aC_f + bC_r}{I_{zz} V_x} \qquad a_{22} = -\frac{a^2 C_f + b^2 C_r}{I_{zz} V_x}$$

$$b_{11} = \frac{C_f}{M_B} \qquad b_{12} = \frac{C_r}{M_B}$$

$$b_{21} = \frac{aC_f}{I_{zz}} \qquad b_{22} = \frac{bC_r}{I_{zz}}$$

$$g_{11} = \frac{1}{aM_B} \qquad g_{12} = 0$$

$$g_{21} = \frac{1}{I_{zz}} \qquad g_{22} = \frac{1}{I_{zz}}$$

3. Optimal Tracking Control

The principles of optimal tracking will be discussed in this section, but restricted an infinite time cost function, and where full state feedback is available. Given a linear time-invariant system of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

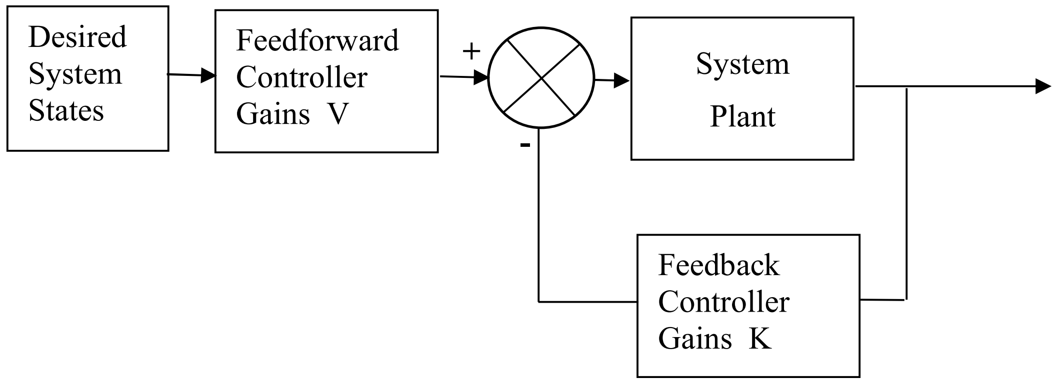

Figure 2. Generic Tracking Control System

The control will be of the form:

$$u(t) = -Kx(t) + Vy_d(t)$$

consisting of a state feedback gain matrix K, and a feedforward gain matrix V. The details of how to obtain the solution are given in the appendix. Figure 2 shows a block diagram of such a system. The feedback gain matrix can be obtained through the solution to the steady state algebraic Ricatti equation:

$$P_1 A + A^T P_1 - P_1 B R^{-1} B^T P_1 + C^T Q C = 0$$

and is given by

$$K = R^{-1} B^T P_1$$

and the feedforward part (to obtain zero steady state error) is given by

$$V = -\left(P_2^T B^T R^{-1} B P_2\right)^{-1} \left(R^{-1} B^T P_2\right)$$

where

$$P_2 = \left(A^T - P_1 B R^{-1} B^T\right)^{-1} C^T Q$$

This suitability scales the feedforward part of the optimal controller in the original development to leave zero steady state error when tracking a constant demand signal.

4. Application to the Vehicle Yaw Moment Control

The two-state vehicle discussed in section 2 needs to be converted into a suitable form to apply the above theory. This can be achieved by incorporating the control terms together, and separating the yaw moment and the augmented steer terms:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \theta_f \\ \Delta\Psi_{Braking} \end{bmatrix}$$

where the A matrix terms are as before and the B matrix terms are

$$b_{11} = \frac{C_f}{M_B} \qquad b_{12} = 0$$

$$b_{21} = \frac{aC_f}{I_{zz}} \qquad b_{22} = \frac{1}{I_z}$$

and $\theta_f = \delta_f + \Delta\delta_f$.

For simplicity we shall consider yaw rate tracking control only, with the desired value given by the steady state relationship between yaw rate and steer angle, as given below:

$$r_d = G_{yaw} \cdot \delta$$

where $G_{yaw} = \dfrac{V_x}{\left(a + b + K_u V_x^2\right)}$ and $K_u = \dfrac{\left(bC_r - aC_f\right)M}{(a+b)C_f C_r}$

In the case $c = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and Q is a scalar. The control matrix is

$$R = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

which enables the control authority of steering and braking terms to be defined.

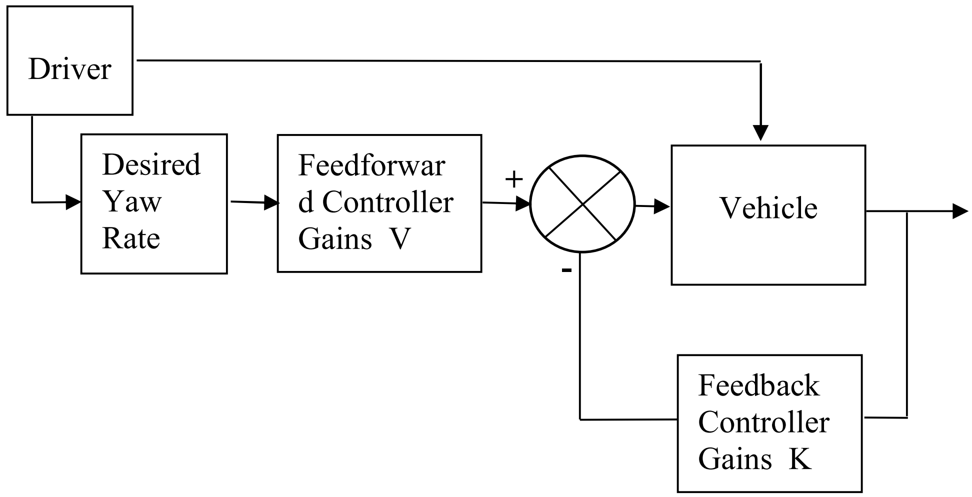

Figure 3. Vehicle Implementation

5. Vehicle Simulation Study

Verification of the above system was performed using CARSIM [CARSIM 2003], a standard commercial vehicle modeling environment that integrates with Matlab/Simulink. The vehicle model used has the parameters given in Table 1 that are typical of a large passenger car. The controls in this case were applied in discrete time intervals of 10ms.

Vehicle Parameter	**Value**
C_f	103109 N/rad
C_r	174823 N/rad
a	1.305 m
b	1.58 m
M_B	1528 Kg
I_{zz}	3132 $Kg \cdot m^2$
t	1.53 m

Table 1. Vehicle Parameter Values

Simulations were performed comparing the performance of the integrated approach developed here to a non-integrated approach where the active front steer and brake control were designed and controlled independently. Figure 4 shows the vehicle performing a double lane change on a road with a surface friction coefficient $\mu = 0.5$. The vehicle with non-integrated systems uses much more braking than is required by the vehicle with integrated systems which maintains its speed better. In the non-integrated approach it can be seen that the vehicle tends to oscillate at the end of the maneuver. This is because the systems were designed to work independently and without consideration of each other. The integrated system also requires less steer angle from the driver, but this is at the expense of greater vehicle sideslip.

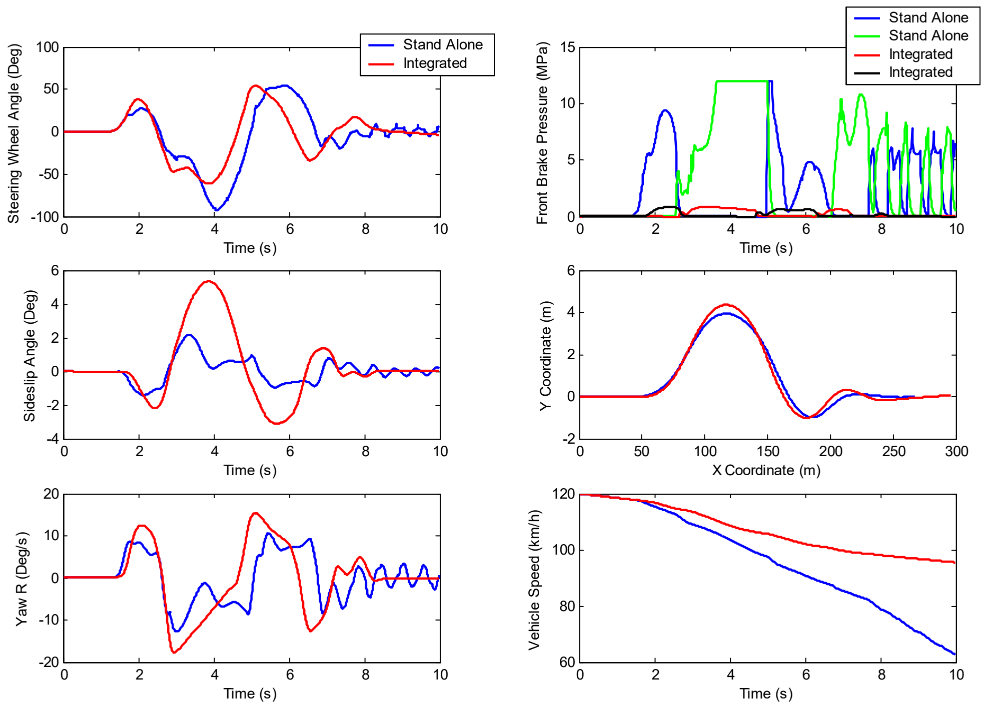

Figure 4. Simulation Results of a double lane change at 120km/h on a $\mu = 0.5$ surface.

Figure 5 shows the simulation results for the vehicles with integrated and non-integrated systems for a double lane change maneuver at 90Km/h on a more slippery surface, with a surface friction coefficient $\mu = 0.35$. For this maneuver, the vehicle without stability control of some form looses control. The simulations show that the non-integrated approach saturates the brakes. If we extend the simulation for another 10 seconds as shown in figure 6, then the non integrated approach eventually results in vehicle instability as the two systems start competing against each other.

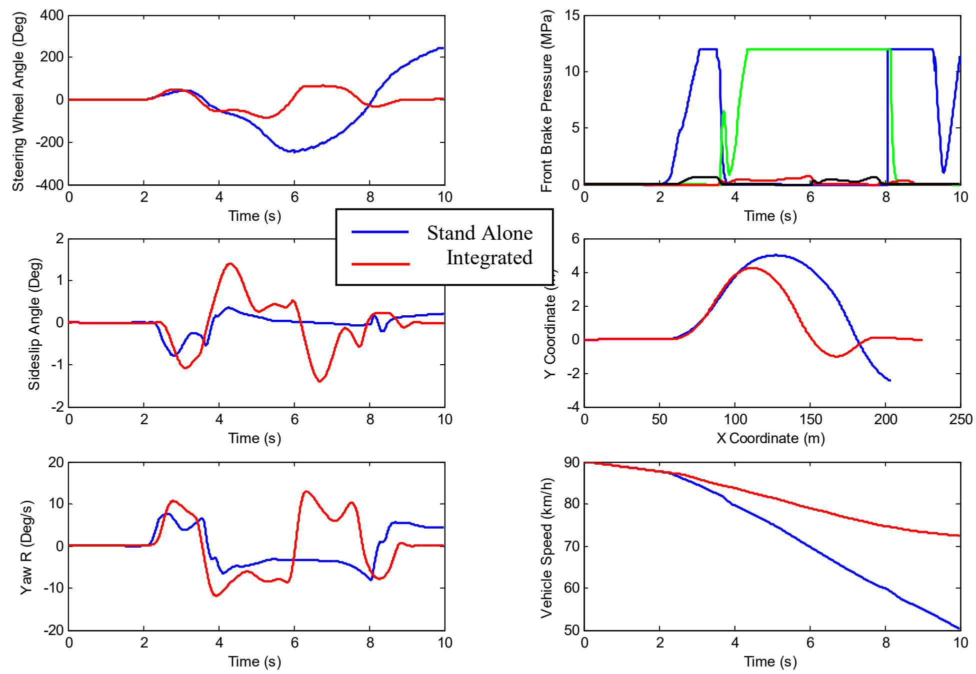

Figure 5. Simulation Results of a double lane change at 90km/h on a $\mu = 0.35$ surface.

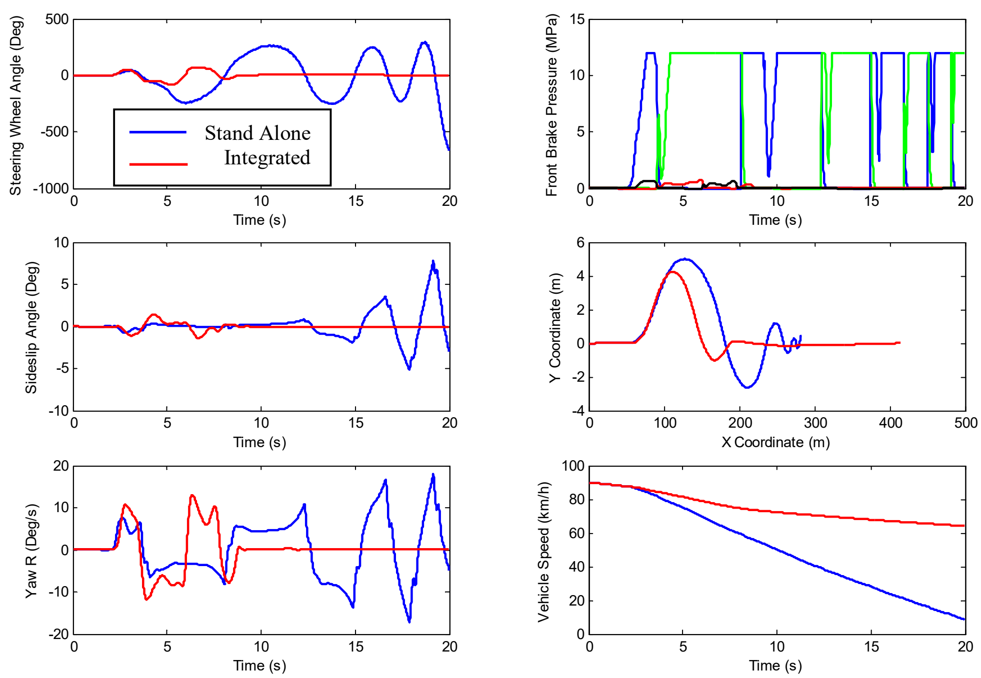

Figure 6. Simulation Results of a double lane change at 90km/h on a $\mu = 0.35$ surface.

With just braking alone or just active front steer, the systems are indeed capable of stabilizing the vehicle in this instance. It is the unintended interactions between these systems that results in the vehicle instability. With active front steer only, a much high level of steer angle is required, and with braking only, the brakes will again saturate.

6. Discussion and Conclusions

This paper has demonstrated the success of applying optimal tracking control for an enhanced vehicle stability control system. The approach enables the integration of multiple actuation systems, here active front steering and braking. The approach is a compromise between applying full nonlinear optimal control and using a state dependent Ricatti equation. The yaw plane models dependence on vehicle speed results in a time varying model and so the Ricatti equation either has to be solved online, at each time instant or tables used with pre-calculated values from which the required value can be interpolated.

The relative trades-off between different actuators can be controlled though the associated cost matrices Q and R. This can be used to adjust the relative proportioning between the different controls. For instance, the cost of using the active front steering can be increased relative to the braking as the steer angle increases. This is a simple way of altering the relative control effort between active front steer and braking before actuator saturation occurs.

While the system discussed here used a simple linear two-state model, more complicated models can be used [Rodić 2000]; however, the simple model has the benefit that the matrix inversion and the Ricatti equation solution can be performed relatively easily. For more complicated models this will not be the case. The inclusion of additional actuation such as controllable dampers, active front and rear roll bars, and active rear steer, enables many more ways to achieve the desired vehicle response. The control approach that is developed here can still applicable. This opens up the possibility of having different driving characteristics available for the vehicle to meet different requirements. The desired vehicle characteristics could be defined by choosing the parameters $C_f, C_r, I_{zz}, M_B, a, b$ for the model:

$$\begin{bmatrix} \dot{V}_{yd} \\ \dot{r}_d \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{C_f + C_r}{M_B V_x}\right) & -\left(\dfrac{aC_f - bC_r}{M_B V_x}\right) - V_x \\ -\left(\dfrac{aC_f - bC_r}{I_{zz} V_x}\right) & \dfrac{a^2 C_f + b^2 C_r}{I_{zz} V_x} \end{bmatrix} \begin{bmatrix} V_{yd} \\ r_d \end{bmatrix} + \begin{bmatrix} \dfrac{C_f}{M_B} \\ \dfrac{aC_f}{I_{zz}} \end{bmatrix} \delta_f$$

This could be applied, for instance, to give a different desired understeer coefficient than the true vehicle parameters dictate. This will alter the feel and responsiveness to the driver, which could also be chosen though a driver mode switching (fun, sporty etc.).

The tracking control aspect has an additional benefit if the system is embedded within a modular control architecture [Gordon 2003]. The tracking aspect can be exploited by other systems on the vehicle such as safety systems. For instance an obstacle avoidance algorithm can use the tracking algorithm to force the vehicle to follow a desired path around objects.

Reference:

C.M. Farmer, "Effects of Electronic Stability Control on Automotive Crash Risk", Traffic Injury Prevention.5:317-325, 2004.

Y.A. Ghoneim, W.C. Lin, D.M. Sidlosky, H.H. Chen, Y-K Chin, and M.J. Tedrake, "Integrated Chassis Control System to Enhance Vehicle Stability", *Int. J. of Vehicle Design*, Vol. 23, Nos. 1/2, 2000.

"OnStar And StabiliTrak To Become Standard Equipment On GM Vehicles", *G.M. Press Release*, January 30, 2005.

M.N. Howell, G.P. Frost, T.J. Gordon and Q.H. Wu, "Continuous action reinforcement learning applied to vehicle suspension control", *Mechatronics*, Vol. 7, No. 3, 263-276, 1997.

G.P. Frost, T.J. Gordon, M.N. Howell and Q.H. Wu, "Moderated reinforcement learning of active and semi-active vehicle suspension control laws", *Proc. Instn. Mech. Engrs. Part I*, 1996, Vol. 210, 249-257.

C. Jun, "The study of ABS control systems with different control methods", 4th *International Symposium on Advanced Vehicle Control AVEC'98*, 623:628, Nagoya, Japan, 1998.

I. Petersen , T. A. Johansen, J. Kalkkuhl and J. Ludemann, "Wheel slip control in ABS using gain scheduled constrained LQR", ECC 2001, Porto.

T. A. Johansen, J. Kalkkuhl, J. Ludemann, I Petersen, "Hybrid Control Strategies in ABS", ACC 2001, Arlington, USA.

RD Fruechte, AM Karmel, JH Rillings, NA Schilke, N.M. Boustany, B.S. Repa, "Integrated Vehicle Control," *IEEE 39th Vehicular Technology Conference,* v 2, pp. 868-877, ISSN : 0740-0551. 1989.

G. Mastinu, E. Babbel, P. Lugner, D. Margolis, P. Mittermayr, B. Richter, "Integrated Controls of Lateral Vehicle Dynamics", *Vehicle System Dynamics*, vol. 23, pp. 358-377, 1993.

G. Roppenecker, H. Wallentowitz, "Integration of Chassis and Traction Control Systems. What Is Possible - What Makes Sense - What Is under Development," *Vehicle System Dynamics*, pp. 283-298, 1993.

M. Tomizuka, J.K. Hedrick, "Advanced Control Methods for Automotive Applications," *Vehicle System Dynamics*, vol. 24, pp. 449-468, 1995.

S.A. Nobe, F.-Y. Wang, "An Overview of Recent Developments in Automated Lateral and Longitudinal Vehicle Controls," in *Proceedings of the IEEE International Conference on Systems, Man and Cybernetics*, vol. 5, pp. 3447-3452, 2001.

H. Smakman, Functional Integration of Active Suspension with Slip Control for Improved Lateral Vehicle Dynamics, *5th International Symposium on Advanced Vehicle Control* (AVEC), Ann Arbor, USA, August 2000, pp 429-436.

M. Harada, H. Harada, Analysis of lateral stability with integrated control of suspension and steering systems, *JSAE Review* 20, pp 465-470, 1999.

Y. Hirano, H. Harada, E. Ono, K. Takanami, "Development of an integrated system of 4WS and 4WD by H∞ control", *SAE International Congress and Exposition*, Detroit, Michigan, USA Mar. 1-5, 1993. SAE-930267.

M. Salman, "Coordinated Control of Braking and Steering," *ASME Symposium on Advanced Automotive Technologies*, 1990 Winter Annual Meeting, AMD-Vol. 108, pp. 69-76.

M. Nagai, Y. Hirano, S. Yamanaka, "Integrated Robust Control of Active Rear Wheel Steering and Direct Yaw Moment Control," *Vehicle System Dynamics*, Supplement 28, pp. 416-421, 1998.

M. Nagai, M. Shino, F. Gao, "Study on Integrated Control of Active Front Steer Angle and Direct Yaw Moment," *JSAE Review*, vol. 23, pp. 309-315, 2002.

S. Sato, H. Inoue, M. Tabata, S. Inagaki, (1992) "Integrated Chassis Control System for Improved Vehicle Dynamics," *International Symposium on Advanced Vehicle Control*, AVEC '92.

S. Horiuchi, K. Okada, S. Nohtomi, "Improvement of Vehicle Handling by Nonlinear Integrated Control of Four Wheel Steering and Four Wheel Torque," *JSAE*, vol. Review 20, pp. 459-464, 1999.

Best M.C. and Gordon T.J., 'Combined state and parameter estimation of vehicle handling dynamics' *5th International Symposium on Advanced Vehicle Control* (AVEC), Ann Arbor, USA, pp 429-436, August 2000,.

A.Y. Ungoren, H. Peng and H. Tseng, "Experimental Verification of Lateral Speed Estimation Methods," Proceedings of the Advanced Vehicle Control Conference, Hiroshima, Japan, September 2002.

Ryu, J., Rossetter, E. J., and Gerdes, J. C., "Vehicle Sideslip and Roll Parameter Estimation Using GPS", P*roceedings of the International Symposium on Advanced Vehicle Control (AVEC)*, 2002, Hiroshima, Japan.

T.D. Gillespie, *Fundamentals of Vehicle Dynamics*, 1st ed. Warrendale, 1992, PA, USA: SAE.

U. Kiencke, L. Nielsen, *Automotive Control Systems*, 1st ed: Springer, 2000.

CARSIM User manual (2005). http://www.carsim.com/

A.D. Rodić, M.K., Vukobratović, Contribution to the Integrated Control Synthesis of Road Vehicles, *IEEE Trans. on Control Systems Technology*, Vol. 7. No. 1, Jan. 1999.

T.J. Gordon, M.N. Howell, F.V. Brandao, "Integrated Control Methodologies for Road Vehicles," *Vehicle System Dynamics*, vol. 40, pp. 157-190, 2003.

Appendix

Optimal Tracking Controller Derivation

Assuming the availability of full state feedback and using linear systems theory

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{A.1}$$

And with a desired tracking output $y_d(t)$. The control can be written as

$$\mathrm{u}(t) = -\mathrm{Kx}(t) + \mathrm{Vr}(t) \tag{A.2}$$

Consider the system

$$\begin{aligned} \dot{\mathrm{x}}(t) &= \mathrm{Ax}(t) + \mathrm{Bu}(t) + \mathrm{G}\,\mathrm{y_d}(t) \\ \mathrm{y}(t) &= \mathrm{Cx}(t) \end{aligned} \tag{A.3}$$

this can be written in the form

$$\dot{\mathrm{x}}(t) = \mathrm{Ax}(t) + \mathrm{B'u'}(t) \tag{A.4}$$

where $\mathrm{B'} = [\mathrm{B} \quad \mathrm{G}]$ and $\mathrm{u}(t)' = [\mathrm{B} \quad \mathrm{y_d}]$

However we can also rewrite it as

$$\dot{\hat{\mathrm{x}}}(t) = \hat{\mathrm{A}}\hat{\mathrm{x}}(t) + \hat{\mathrm{B}}\mathrm{u}(t) \tag{A.5}$$

where

$$\hat{A} = \begin{bmatrix} \mathrm{A} & \mathrm{G} \\ 0 & \mathrm{A_0} \end{bmatrix} \qquad \hat{\mathrm{B}} = \begin{bmatrix} \mathrm{B} \\ 0 \end{bmatrix} \qquad \hat{\mathrm{x}} = \begin{bmatrix} \mathrm{x} \\ y_d \end{bmatrix}$$

Where the reference input is considered here, for the purpose of controller design and development as a pseudo-state with a small time constant. The function to be minimized is the cost function

$$J = \int_0^\infty \left\{ (\mathrm{y_d}(t) - \mathrm{y}(t))^{\mathrm{T}} \mathrm{Q} (\mathrm{y_d}(t) - \mathrm{y}(t)) + \mathrm{u}(t)^{\mathrm{T}} \mathrm{Ru}(t) \right\} dt \tag{A.6}$$

with the system defined in this new manner we can let

$$\mathrm{y_d} = \mathrm{F}\hat{\mathrm{x}} \text{ and } \mathrm{y} = \hat{\mathrm{C}}\hat{\mathrm{x}}$$

where

$$F = [0 \quad \dots \quad 0 \quad 1] \text{ and } \hat{\mathrm{C}} = [C \quad 0]$$

$$J = \int_0^\infty \left\{ \hat{\mathrm{x}}^{\mathrm{T}}(t) \hat{\mathrm{Q}} \hat{\mathrm{x}}(t) + \mathrm{u}(t)^{\mathrm{T}} \mathrm{Ru}(t) \right\} dt \tag{A.7}$$

which is the ‘standard form’ so the results from optimal regulation can be applied

$$\begin{aligned} u^*(t) &= -R^{-1}\hat{B}^T P\hat{x}(t) \\ &= -R^{-1}\begin{bmatrix} B & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} x \\ y_d \end{bmatrix} \\ &= -R^{-1}B^T P_1 x(t) - R^{-1}B^T P_2 y_d(t) \end{aligned} \tag{A.8}$$

$$\begin{aligned} &\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} A & G \\ 0 & A_0 \end{bmatrix} + \begin{bmatrix} A^T & 0 \\ G^T & A_0^T \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \\ &-\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} B^T \\ 0 \end{bmatrix} R^{-1}\begin{bmatrix} B^T & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} + \begin{bmatrix} C^T QC & -C^T Q \\ -QC & Q \end{bmatrix} = 0 \end{aligned} \tag{A.9}$$

Expanding

$$\begin{aligned} &\begin{bmatrix} P_1 A & P_2 G + P_2 A_0 \\ P_2^T A & P_2^T G + P_3 A_0 \end{bmatrix} + \begin{bmatrix} A^T P_1 & A^T P_2 \\ G^T P_1 + A_0^T P_2^T & G^T P_2 + A_0^T P_3 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \\ &-\begin{bmatrix} P_1 BR^{-1}B^T P_1 & P_1 BR^{-1}B^T P_2 \\ P_2 BR^{-1}B^T P_1 & P_2^T BR^{-1}B^T P_2 \end{bmatrix} + \begin{bmatrix} C^T QC & -C^T Q \\ -QC & Q \end{bmatrix} = 0 \end{aligned} \tag{A.10}$$

which gives

$$\begin{aligned} &P_1 A + A^T P - P_1 BR^{-1}B^T P_1 + C^T QC = 0 \\ &P_1 G + P_2 A_0 + A^T P_2 - P_1 BR^{-1}B^T P_2 - C^T Q = 0 \\ &P_2^T A + G^T P_1 + A_0^T P_2^T - P_2^T BR^{-1}B^T P_1 - QC = 0 \\ &P_2^T G + P_3 A_0 + G^T P_2 + A_0^T P - P_2^T BR^{-1}B^T P_2 + Q = 0 \end{aligned} \tag{A.11}$$

P_1 can easily be obtained from the first of these equations which is equivalent to the Ricatti equation.

P_2 can be obtained from the second we can set $A_0 = 0$ and note that here that $G = 0$.

$$P_2 = \left(A^T - P_1 BR^{-1}B^T\right)^{-1} C^T Q = 0 \tag{A.12}$$

For zero steady state offset consider the system at steady state i.e. $\dot{\mathbf{x}} = 0$.

$$\dot{x}(t) = Ax(t) + Bu(t) = 0 \tag{A.13}$$

where

$$u(t) = -R^{-1}B^T P_1 x(t) + Vy_d(t) \tag{A.14}$$

so

$$\mathrm{x}(t) = -\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{-1}\mathrm{BVy}_{\mathrm{d}}(t) \tag{A.15}$$

Applying the following equalities

$$\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{T} = \mathrm{A}^{\mathrm{T}} - \mathrm{P}_1\mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1 \tag{A.16}$$

and

$$\left(\mathrm{A}^{T}\right)^{-1} = \left(\mathrm{A}^{-1}\right)^{T} \tag{A.17}$$

and with

$$\mathrm{P}_2 = \left(\mathrm{Q}^{\mathrm{T}}\mathrm{C}\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{-1}\right)^{T} \tag{A.18}$$

With zero steady state error $\mathrm{y}_{\mathrm{d}}(t) = \mathrm{Cx}$, so

$$\mathrm{y}_{\mathrm{d}}(t) = \mathrm{Cx}(t) = -\mathrm{C}\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{-1}\mathrm{BVy}_{\mathrm{d}}(t) \tag{A.19}$$

$$\mathrm{I} = -\mathrm{C}\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{-1}\mathrm{BVy}_{\mathrm{d}}(t) \tag{A.20}$$

Multiplying by Q^{T} gives

$$\begin{aligned}\mathrm{Q}^{T} &= -\mathrm{Q}^{T}\mathrm{C}\left(\mathrm{A} - \mathrm{BR}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P}_1\right)^{-1}\mathrm{BVy}_{\mathrm{d}}(t) \\ \mathrm{Q}^{T} &= -\mathrm{P}_2^{T}\mathrm{BV}\end{aligned} \tag{A.21}$$

or

$$\mathrm{Q} = -\mathrm{VB}^{\mathrm{T}}\mathrm{P}_2 \tag{A.22}$$

Note, Q will be a scalar if we are controlling to a single reference signal. V can be obtained then using the pseudo inverse,

$$\mathrm{V} = -\mathrm{Q}\left(\mathrm{B}^{\mathrm{T}}\mathrm{P}_2\right)^{+} \tag{A.23}$$

The pseudo inverse, $\mathrm{B}^{+} = \left(\mathrm{B}^{T}\mathrm{B}\right)^{-1}\mathrm{B}^{T}$, is obtain from $\mathrm{Bz} = \mathrm{C}$ by pre-multiplying both sides by B^{T} to obtain a square matrix, the in the inverse of $\left(\mathrm{B}^{T}\mathrm{B}\right)$ exists then

$$\mathrm{B}^{T}\mathrm{Bz} = \mathrm{B}^{T}\mathrm{C} \tag{A.24}$$

$$\begin{aligned} z &= \left(B^T B\right)^{-1} B^T C \\ &= B^+ C \end{aligned} \tag{A.25}$$

Lemma

For zero steady state error

$$V = -R^{-1} B^T P_2 Q \left(P_2^T B R^{-1} B^T P_2\right)^{-1}$$

To obtain the above equation (equation (1.19) in the main text) starting from equation (A.21)

$$Q^T = -P_2^T B V$$

where let $V = V_1 V_2$ and $V_1 = R^{-1} B^T P_2$ from the original optimal control formulation, equation (1.6).

$$\begin{aligned} Q^T &= -P_2^T B V \\ &= -P_2^T B V_1 V_2 \\ &= -P_2^T B R^{-1} B^T P_2 V_2 \end{aligned}$$

Note $\left(P_2^T B R^{-1} B^T P_2\right) = \left(P_2^T B R^{-1} B^T P_2\right)^T$, so

$$Q = -V_2 \left(P_2^T B R^{-1} B^T P_2\right)$$

rearranging gives

$$V_2 = -Q \left(P_2^T B R^{-1} B^T P_2\right)^{-1}$$

Therefore the total feedforward part V is given by

$$V = -R^{-1} B^T P_2 Q \left(P_2^T B R^{-1} B^T P_2\right)^{-1}$$

□.